1 Homework 11

\[ \binom{n}{k} = \binom{n}{n-k} \]

Formula for the mean:

\[ \mathbb{E}(X) = \sum x P(X = x) \]

Formula for the variance:

\[ \text{var}(X) = \mathbb{E}[(X - \mu)^2] \]
\[ = \sum (x - \mu)^2 P(X = x) \]

Another way to compute the variance:

\[ \text{var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2. \]  \hspace{1cm} (1)

Using the above formula for the mean, we’d have:

\[ \mathbb{E}(X^2) = \sum x^2 P(X = x) \]

Formula for the standard deviation:

\[ \sigma = \sqrt{\text{var}(X)} \]

1.1 Exercise 4

Roll a fair die twice. \( X \) is the random variable that gives the maximum of the two numbers. Find the probability mass function of \( X \).

Let’s first make a chart of the outcomes. We have
and so we have a table for the probability mass function as follows

<table>
<thead>
<tr>
<th>$x$</th>
<th>$P(X = x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/36</td>
</tr>
<tr>
<td>2</td>
<td>3/36</td>
</tr>
<tr>
<td>3</td>
<td>5/36</td>
</tr>
<tr>
<td>4</td>
<td>7/36</td>
</tr>
<tr>
<td>5</td>
<td>9/36</td>
</tr>
<tr>
<td>6</td>
<td>11/36</td>
</tr>
</tbody>
</table>

### 1.2 Exercise 6

An urn contains five green, two blue, and three red. Remove three without replacement. Let $X$ denote the number of red balls.

We need to find the probability that there are 0, 1, 2, and 3 red balls.

#### 1.2.1 Method 1

The probability that there are no red balls pulled is

$$\frac{7}{10} \cdot \frac{6}{9} \cdot \frac{5}{8}.$$

The probability that exactly one red ball is pulled is

$$\frac{3}{10} \cdot \frac{7}{9} \cdot \frac{6}{8} + \frac{7}{10} \cdot \frac{3}{9} \cdot \frac{6}{8} + \frac{7}{10} \cdot \frac{6}{9} \cdot \frac{3}{8} = 3 \cdot \frac{7}{10} \cdot \frac{6}{9} \cdot \frac{3}{8},$$

where the first term gives the probability of pulling out the red as the first ball, the second term as the second ball, and third term as the third ball.

Similarly, the probability that exactly two red balls are pulled is

$$3 \cdot \frac{7}{10} \cdot \frac{3}{9} \cdot \frac{2}{8}.$$
And the probability that exactly three red balls are pulled is
\[
\frac{3}{10} \cdot \frac{2}{9} \cdot \frac{1}{8}.
\]

1.2.2 Method 2

Alternatively, the total combinations of three balls removed without replacement is \(\binom{10}{3}\). To get 0 red we have \(\binom{7}{3}\) combinations. For exactly 1 red we have \(\binom{7}{2}\) \(\binom{3}{1}\). For exactly two red we have \(\binom{7}{1}\) \(\binom{3}{2}\). For exactly three red we have \(\binom{3}{3}\). Taking the quotients, we get the same values as above.

1.2.3 Solution

\[
\begin{array}{|c|c|}
\hline
x & P(X = x) \\
\hline
0 & 210/720 \\
1 & 378/720 \\
2 & 126/720 \\
3 & 6/720 \\
\hline
\end{array}
\]

1.2.4 Check Solution

If we counted correctly, these numbers should add up. From the final solution or Method 1, we have \(210 + 378 + 126 + 6 = 720\). Good. From Method 2 we have \(35 + 63 + 21 + 1 = 120\).

1.3 Exercise 14a

Let \(Y\) be a random variable that counts the number of trials until the first heads shows up, where the probability of heads is \(p\).

- \(P(Y = 1)\) is the probability that it takes one coin flip to obtain a heads on the first trial. This is exactly \(p\).
- \(P(Y = 2)\) is the probability that it takes two coin flips until a heads appears. Thus the first trial must come up a tails and the second comes up heads. The two coin flips are independent events, so we multiply their probability. We have \((1 - p)\) for the first trial and \(p\) for the second. We conclude 
  \[P(Y = 2) = (1 - p)p.\]

Similarly, \(P(Y = 3)\) requires two flips of tails and then a flip of heads and so we have corresponding probabilities of \((1 - p)\), \((1 - p)\), and \(p\). The events are independent so we multiply to obtain 
  \[P(Y = 3) = (1 - p)^2 p.\]

Remark 1: One notices a pattern and part b of exercise 14 says that
  \[P(Y = j) = (1 - p)^{j-1} p.\]

Remark 2: Part c of exercise 14 claims the sum of these infinite events is 1 by defining an infinite sum as a limit of partial sums.
1.4 Exercise 22

Given the following probability mass function

<table>
<thead>
<tr>
<th>$x$</th>
<th>$P(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0.1</td>
</tr>
<tr>
<td>-0.5</td>
<td>0.2</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>0.5</td>
<td>0.25</td>
</tr>
<tr>
<td>1</td>
<td>0.35</td>
</tr>
</tbody>
</table>

Compute the mean:

$$E(X) = (-1)(0.1) + (-0.5)(0.2) + (0.1)(0.1) + (0.5)(0.25) + (1)(0.35)$$
$$= -0.1 + (-0.1) + 0.01 + 0.125 + 0.35$$
$$= 0.285$$

Compute the variance. We already computed $E(X)$, so let's compute $E(X^2)$ and use formula 1 on the first page. We have

$$E(X^2) = (1)(0.1) + (0.25)(0.2) + (0.01)(0.1) + (0.25)(0.25) + (1)(0.35)$$
$$= 0.1 + 0.05 + 0.001 + 0.0625 + 0.35$$
$$= 0.5635$$

Thus,

$$\text{var}(X) = E(X^2) - [E(X)]^2$$
$$= 0.5635 - (0.285)^2$$
$$= 0.482275$$

And so the standard deviation is

$$\sigma = \sqrt{\text{var}(X)} \approx 0.694460$$

Looking back at our probability distribution function, we could say this answer makes sense and is at least on the right order of magnitude.

1.5 Exercise 24

Let $X$ be uniformly distributed on the set $S = \{1, 2, 3, \ldots, n\}$ where $n$ is a positive integer. In other words, $X$ has the probability mass function

$$P(X = k) = \frac{1}{n} \text{ for any } k \in S.$$  

For a concrete example of this problem, do Exercise 23.
We have

\[ E(X) = \sum_{x \in S} xP(X = x) \]
\[ = \sum_{k=1}^{n} kP(X = k) \]
\[ = \sum_{k=1}^{n} k \cdot \frac{1}{n} \]
\[ = \frac{1}{n} \sum_{k=1}^{n} k \]
\[ = \frac{1}{n} \frac{n(n+1)}{2} \]
\[ = \frac{n+1}{2} \]

and we also have

\[ E(X^2) = \sum_{x \in S} x^2P(X = x) \]
\[ = \sum_{k=1}^{n} k^2P(X = k) \]
\[ = \frac{1}{n} \sum_{k=1}^{n} k^2 \]
\[ = \frac{1}{n} \frac{n(n+1)(2n+1)}{6} \]
\[ = \frac{(n+1)(2n+1)}{6} \]

Then

\[ \text{var}(X) = E(X^2) - [E(X)]^2 \]
\[ = \frac{(n+1)(2n+1)}{6} - \left( \frac{n+1}{2} \right)^2 \]
\[ = \frac{2n^2 + 3n + 1}{12} - \frac{n^2 + 2n + 1}{4} \]
\[ = \frac{n^2 - 1}{12} \]

1.6 Exercise 30

The exercise provides us with the probability mass function for independent random variables \( X \) and \( Y \).
Then we can compute $E(X)$ and $\text{var}(X)$ using the following table:

<table>
<thead>
<tr>
<th>$k$</th>
<th>$P(X = k)$</th>
<th>$P(Y = k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>-1</td>
<td>0.1</td>
<td>0.2</td>
</tr>
<tr>
<td>0</td>
<td>0.2</td>
<td>0.1</td>
</tr>
<tr>
<td>0.5</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>2</td>
<td>0.15</td>
<td>0.1</td>
</tr>
<tr>
<td>2.5</td>
<td>0.15</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Then we can compute $E(X)$ and $\text{var}(X)$ using the following table:

<table>
<thead>
<tr>
<th>$P(X = k)$</th>
<th>$k$</th>
<th>$kP(X = k)$</th>
<th>$k^2$</th>
<th>$k^2P(X = k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-3</td>
<td>-0.3</td>
<td>9</td>
<td>0.9</td>
</tr>
<tr>
<td>0.1</td>
<td>-1</td>
<td>-0.1</td>
<td>1</td>
<td>0.1</td>
</tr>
<tr>
<td>0.2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.3</td>
<td>0.5</td>
<td>0.15</td>
<td>0.25</td>
<td>0.075</td>
</tr>
<tr>
<td>0.15</td>
<td>2</td>
<td>0.3</td>
<td>4</td>
<td>0.6</td>
</tr>
<tr>
<td>0.15</td>
<td>2.5</td>
<td>0.375</td>
<td>6.25</td>
<td>0.937</td>
</tr>
</tbody>
</table>

$E(X) = 0.43$  
$E(X^2) = 2.61$

It follows from the table that $\text{var}(X) = 2.61 - 0.43^2 \approx 2.43$.

Similarly we can compute $E(Y)$ and $\text{var}(Y)$ using the following table:

<table>
<thead>
<tr>
<th>$P(Y = k)$</th>
<th>$k$</th>
<th>$kP(Y = k)$</th>
<th>$k^2$</th>
<th>$k^2P(Y = k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-3</td>
<td>-0.3</td>
<td>9</td>
<td>0.9</td>
</tr>
<tr>
<td>0.2</td>
<td>-1</td>
<td>-0.2</td>
<td>1</td>
<td>0.2</td>
</tr>
<tr>
<td>0.1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.3</td>
<td>0.5</td>
<td>0.15</td>
<td>0.25</td>
<td>0.075</td>
</tr>
<tr>
<td>0.1</td>
<td>2</td>
<td>0.2</td>
<td>4</td>
<td>0.4</td>
</tr>
<tr>
<td>0.2</td>
<td>2.5</td>
<td>0.5</td>
<td>6.25</td>
<td>1.25</td>
</tr>
</tbody>
</table>

$E(X) = 0.35$  
$E(X^2) = 2.825$

It follows from the table that $\text{var}(Y) = 2.825 - 0.35^2 \approx 2.70$.

Then we have $E(X + Y) = E(X) + E(Y) = 0.40 + 0.35 = 0.75$ and $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y) \approx 2.43 + 2.70 = 5.13$. 

6
2 Homework 12

Binomial distribution. Number of independent trials is \( n \), number of successes is \( k \), where the probability of a success is \( p \).

\[
P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}
\]

Also important is knowing the steps to computing

\[
\lim_{n \to \infty} \left(1 + \frac{a}{n}\right)^n
\]

or similar. See Exercise 70 for an example and page 713 for another example.

We have

\[
\lim_{n \to \infty} \exp \left[ n \ln \left(1 + \frac{a}{n}\right) \right] = \exp \left[ \lim_{n \to \infty} n \ln \left(1 + \frac{a}{n}\right) \right].
\]

And we have

\[
\lim_{n \to \infty} n \ln \left(1 + \frac{a}{n}\right) = \lim_{n \to \infty} \frac{\ln (1 + \frac{a}{n})}{\frac{1}{n}}.
\]

But since

\[
\lim_{y \to \infty} \frac{\ln (1 + \frac{a}{y})}{\frac{1}{y}} = \lim_{y \to \infty} \frac{\frac{1}{1 + \frac{a}{y} - 1}}{\frac{-a}{y^2}} = \lim_{y \to \infty} \frac{a}{1 + \frac{a}{y}} = a
\]

by an application of l’Hospital’s rule, we determine that

\[
\lim \left(1 + \frac{a}{n}\right)^n = e^a.
\]

A useful fact to know is that

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}
\]  \hspace{1cm} (2)

2.1 Exercise 34

Toss a coin (with probability of heads equal to 0.3) five times. Let \( X \) be the number of tails. What is \( P(X = 2) \) and \( P(X \geq 1) \)?

Coin tosses are independent, so we have \( P(X = 2) = \binom{5}{2}(0.7)^2(0.3)^3 \). To compute \( P(X \geq 1) \) it’s easier to just compute the probability of the complement \( P(X = 0) \) and remember to subtract that answer from 1. We have \( P(X = 0) = \binom{5}{0}(0.7)^0(0.3)^5 = (0.3)^5 \) and so \( P(X \geq 1) = 1 - (0.3)^5 \).
2.2 Exercise 38

Four green and six blue balls. Draw a ball at random. Note its color. Replace it. Repeat these steps four times. Let \( X \) denote the total number of green balls you obtain. Find the probability mass function of \( X \). Replacement means the trials are independent. Thus we have the following table:

<table>
<thead>
<tr>
<th>( k )</th>
<th>( P(X = k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \binom{4}{0} (.4)^0 (.6)^4 )</td>
</tr>
<tr>
<td>1</td>
<td>( \binom{4}{1} (.4)^1 (.6)^4 )</td>
</tr>
<tr>
<td>2</td>
<td>( \binom{4}{2} (.4)^2 (.6)^4 )</td>
</tr>
<tr>
<td>3</td>
<td>( \binom{4}{3} (.4)^3 (.6)^4 )</td>
</tr>
<tr>
<td>4</td>
<td>( \binom{4}{4} (.4)^4 (.6)^4 )</td>
</tr>
</tbody>
</table>

2.3 Exercise 40

Four red, seven green, two white. Draw, note, replace. Repeat four times. Let \( X \) denote the number of red balls and \( Y \) the number of green balls. Find \( P(X + Y = 2) \).

*The book erroneously writes \( P(X + Y) = 2 \).

2.3.1 Method 1

\( X + Y = 2 \) occurs when the number of red and green balls is exactly two. The probability of being either a red or green ball is 11 out of 13. Replacement means the trials are independent. We have

\[
P(X + Y = 2) = \binom{4}{k} p^k (1-p)^{4-k}
\]

\[
= \binom{4}{2} \left( \frac{11}{13} \right)^2 \left( \frac{2}{13} \right)^2 = \frac{2904}{28561} = \frac{2904}{13^4}.
\]

2.3.2 Method 2

\( X + Y = 2 \) means the number of white balls is always two. Let \( Z \) denote the number of white balls. Then

\[
P(X + Y = 2) = P(Z = 2)
\]

\[
= \binom{4}{2} \left( \frac{2}{13} \right)^2 \left( \frac{11}{13} \right)^2 = \frac{2904}{13^4}.
\]
2.3.3 Method 2 (Uses 12.4.4 - The Multinomial Distribution)  
( Longer, Still Good)

\[
P(X + Y = 2) = P(X = 0, Y = 2) + P(X = 1, Y = 1) + P(X = 2, Y = 0) \\
= P(X = 0, Y = 2, Z = 2) + P(X = 1, Y = 1, Z = 2) + P(X = 2, Y = 0, Z = 2) \\
= \frac{4!}{0!2!2!} \left( \frac{4}{13} \right)^0 \left( \frac{7}{13} \right)^2 \left( \frac{2}{13} \right)^2 + \frac{4!}{1!1!2!} \left( \frac{4}{13} \right)^1 \left( \frac{7}{13} \right)^1 \left( \frac{2}{13} \right)^2 + \frac{4!}{2!0!2!} \left( \frac{4}{13} \right)^2 \left( \frac{7}{13} \right)^0 \left( \frac{2}{13} \right)^2 \\
= \frac{4!}{2!2!} \frac{4}{13^2} (7^2 + 2 \cdot 4 \cdot 7 + 4^2) \\
= \frac{24}{13^4} (7 + 4)^2 = 2904 \frac{13^4}{13^4}.
\]

2.4 Exercise 42

Prevalence of 1 in 100. 10 individuals pooled and tested. What is the probability that no individual has the disease?

The probability is \( (0.99)^{10} \).

2.5 Exercise 46

A true-false exam has 20 questions. What is the expected number of correct answers if a student guesses the answers at random.

This is a binomial distribution with 20 trials. The probability of a student getting a particular question right at random is 0.5. The expected value is

\[
E(S_n) = np = 20 \cdot 0.5 = 10.
\]

2.6 Exercise 54

Urn contains six green, eight blue, and ten red balls. Take, note, replace. Repeat six times. What is the probability that you sampled two of each color?

This is a multinomial distribution. Let \( N_1 \) be the number of greens, \( N_2 \) be the number of blues, and \( N_3 \) be the number of reds. Then

\[
P(N_1 = a, N_2 = b, N_3 = c) = \frac{(a + b + c)!}{a!b!c!} \left( \frac{6}{24} \right)^a \left( \frac{8}{24} \right)^b \left( \frac{10}{24} \right)^c.
\]

Since we repeat six times, the sum must be 6. In this particular question \( a = b = c = 2 \).

\[
P(N_1 = 2, N_2 = 2, N_3 = 2) = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{2^3} \left( \frac{1}{4} \right)^2 \left( \frac{1}{3} \right)^2 \left( \frac{5}{12} \right)^2 \\
= \frac{125}{1152}
\]
2.7 Exercise 58

Attached earlobe is caused by a pair of recessive genes \((aa)\). Given a couple \((Aa)\) and \((aa)\), what is the probability the child has an unattached earlobe?

Writing out the possibilities, we have

\[
\begin{array}{c|cc}
 & A & a \\
\hline 
a & As & aa \\
a & Aa & aa \\
\end{array}
\]

and conclude the probability is a half.

2.8 Exercise 60

Hemophilia is sex-linked recessive. We mix \((X_{\text{hemo}}X)\) and \((XY)\). Among the two sons and daughters, what is the probability that one daughter is not a carrier, one daughter is a carrier, one son is hemophilic, and one son is not hemophilic.

First we write out the possibilities

\[
\begin{array}{c|c|c|}
 & X_{\text{hemo}} & X \\
\hline 
X & (X_{\text{hemo}}X) & (XX) \\
Y & (X_{\text{hemo}}Y) & (XY) \\
\end{array}
\]

Thus there is probability \(p = 0.5\) that a daughter is a carrier and \(p = 0.5\) that a son is hemophilic. Because there are two daughters and consider the situation for exactly one is a carrier, then the binomial distribution tells us that

\[
p(\text{exactly one daughter is a carrier}) = \binom{2}{1}(0.5)^1(0.5)^1 = 0.5
\]

and similarly

\[
p(\text{exactly one son is hemophilic}) = \binom{2}{1}(0.5)^1(0.5)^1 = 0.5.
\]

To have the two situations occurring together, we multiple and get

\[
p(\text{one daughter is carrier, one is not, one son is hemophilic, one is not}) = 0.25.
\]

2.9 Exercise 62

Coin. \(p = .3\) of heads. Probability first time appears is fifth trial.

Answer: \((.7)^4(.3)\)

2.10 Exercise 68

Rolling a fair die until the first time a 1 or 2 appears. Find the probability that the first 1 or 2 appears within the first five trials.
The probability that a 1 or 2 appears is $\frac{1}{3}$. Note that the probability that it appears within the first five trials is the complement to it not showing within the first five trials. Thus

$$p(1 \text{ or } 2 \text{ doesn’t appear within the first five trials}) = 1 - \left(\frac{2}{3}\right)^5.$$ 

For the unbelievers, we write:

$$p = \frac{1}{3} + \left(\frac{2}{3}\right)^1 \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^4 + \frac{1}{3} \left[ 1 + \left(\frac{2}{3}\right)^1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 \right]$$

$$= \frac{1}{3} - \left(\frac{2}{3}\right)^5 = 1 - \left(\frac{2}{3}\right)^5.$$

### 2.11 Exercise 70

Urn contains one black and $n - 1$ white balls. Balls drawn one at a time randomly until the black ball is selected. Each ball is replaced before the next ball is drawn. Find the probability that at least $n$ draws are needed. What happens as $n \to \infty$?

The probability of drawing a black ball is $\frac{1}{n}$. The event of needing at least $n$ draws happens when white is drawn for the first $n - 1$ draws. Thus the probability is

$$\left(1 - \frac{1}{n}\right)^{n-1}.$$ 

Taking the limit as $n$ goes to infinity, we get the constant $e^{-1}$. As a review, we recall that the limiting case $1^\infty$ is special and can be dealt with by first rewriting

$$\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^{n-1} = \lim_{n \to \infty} \exp \left[ (n-1) \ln \left(1 - \frac{1}{n}\right) \right]$$

$$= \exp \left[ \lim_{n \to \infty} (n-1) \ln \left(1 - \frac{1}{n}\right) \right].$$

The limit on the inside of the exponentiation is of type $\infty \cdot 0$ and can be dealt with by yet again rewriting

$$\lim_{n \to \infty} (n-1) \ln \left(1 - \frac{1}{n}\right) = \lim_{n \to \infty} \frac{\ln \left(1 - \frac{1}{n}\right)}{\frac{1}{n}}.$$
Finally we get a limit of the time \( \frac{9}{5} \). We apply l'Hospital’s rule to the corresponding functions

\[
\lim_{x \to \infty} \frac{\ln \left(1 - \frac{1}{x}\right)}{x - 1} = \lim_{x \to \infty} \frac{\frac{1}{x^2}}{1 - \frac{1}{(x - 1)^2}} = -1
\]

and conclude

\[
\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^{n-1} = \exp[-1] = \frac{1}{e}.
\]

See the beginning of Section 2 for the general case.

### 2.12 Exercise 74

Urn contains one black and \( n - 1 \) white balls. Balls drawn one at a time randomly until the black ball is selected. Each ball is replaced before the next ball is drawn. Find the probability that exactly \( k \) white balls will be drawn before the black one is if (a) each ball is replaced before the next ball is drawn and (b) balls are not replaced.

If the balls are replaced, we have

\[
\left(1 - \frac{1}{n}\right)^k \left(\frac{1}{n}\right).
\]

If the balls are not replaced, we have

\[
\frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \frac{n-3}{n-2} \cdots \frac{n-k}{n-k+1} \cdot \frac{1}{n-k} = \frac{1}{n}.
\]

### 2.13 Exercise 80

Suppose \( X \) is a Poisson distributed with parameter \( \lambda = 0.2 \). Find \( P(X < 3) \) and find \( P(2 \leq X \leq 3) \).

Using the formula, we have

\[
P(X = 1) = e^{-\lambda} \frac{\lambda^1}{1!},
\]

\[
P(X = 2) = e^{-\lambda} \frac{\lambda^2}{2!},
\]

and

\[
P(X = 3) = e^{-\lambda} \frac{\lambda^3}{3!}.
\]

\[
P(X < 3) = P(X = 1) + P(X = 2) \quad \text{and}
\]

\[
P(2 \leq X \leq 3) = P(X = 2) + P(X = 3).
\]
2.14 Exercise 86

Suppose the number of phone calls arriving at a switchboard per hour is Poisson distributed with mean 3 calls per hour. Find the probability that at least one phone call arrives between noon and 1 P.M. Assuming that phone calls in different hours are independent of each other, find the probability that no phone calls arrive between noon and 2 P.M.

Note that the mean tells us the value of $\lambda$, so $\lambda = 3$. Then the probability of at least one phone call in an hour time slot is

$$P(X \geq 1) = 1 - P(X < 1)$$

$$= 1 - P(X = 0)$$

$$= 1 - e^{-\lambda} \frac{\lambda^0}{0!}$$

$$= 1 - e^{-3}.$$

The probability that no phone calls arrive in two hours, due to the independence is the product of probabilities. We have

$$P(X = 0) = e^{-3}$$

and so the probability is $(e^{-3})^2 = e^{-6}$.

2.15 Exercise 94

Suppose $X$ and $Y$ are independent and Poisson with mean $\lambda$. Given that $X + Y = n$, find the probability that $X = k$ for $k = 0, 1, \ldots, n$.

The sum of Poisson random variables with means $\lambda_1$ and $\lambda_2$ is another Poisson distributed with parameter $\lambda_1 + \lambda_2$. The book derives this on page 715. It makes use of 2 on page 7. In any case, we have

$$P(X = k \mid X + Y = n) = \frac{P(X = k \text{ and } X + Y = n)}{P(X + Y = n)}.$$

The numerator is the same as $P(X = k \text{ and } Y = n - k)$. Since $X$ and $Y$ are independent, we have

$$P(X = k \text{ and } Y = n - k) = P(X = k)P(Y = n - k)$$

$$= \left( e^{-\lambda} \frac{\lambda^k}{k!} \right) \left( e^{-\lambda} \frac{\lambda^{n-k}}{(n-k)!} \right)$$

$$= e^{-2\lambda} \frac{\lambda^n}{k!(n-k)!}.$$

Because $X + Y$ is Poisson with mean $2\lambda$, we have

$$P(X + Y = n) = e^{-2\lambda} \frac{(2\lambda)^n}{n!}.$$
and so

\[ P(X = k \mid X + Y = n) = \frac{1}{2^n} \cdot \frac{n!}{k!(n-k)!} = \frac{1}{2^n} \binom{n}{k} \]

2.16 Exercise 96

1 in 500 experience side effects. Using the Poisson approximation, find the probability that in a group of 200 people, at least 1 person experiences side effects.

At first, we have

\[ P(S_{200} \geq 1) = 1 - P(S_{200} = 0) \]

To approximate this with a Poisson distribution, we have the parameter \( \lambda \) which is the average, to be the \( np = 200 \cdot \frac{1}{500} = \frac{2}{5} \). Note that the probability of experiencing a side effect is small, part of what allows us to approximate using a Poisson distribution.

\[ P(X = 0) = e^{-\lambda}. \]

In conclusion, we have

\[ P(S_{200} \geq 1) \approx 1 - e^{-2/5} \]

2.17 Exercise 98

About 1 in 1000 boys is affected by fragile X syndrome, a genetic disorder that causes learning difficulties. Find the probability that, in a group of 500 boys, nobody is affected by this disorder by (a) computing the exact probability and (b) using a Poisson approximation.

The exact probability is

\[ \left( \frac{999}{1000} \right)^{500} \approx 0.60637 \]

and the Poisson approximation is \((\lambda = 500 \cdot \frac{1}{1000} = .5)\)

\[ e^{-\lambda} \approx 0.60653. \]

(I just used a calculator for both.)